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On Polynomials with Positive Coefficients

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All polynomials in this paper are supposed to have real coefficients. Polynomials which can be represented in the form

$$p(x)=\sum_{\substack{k+l\leqslant m,\\ k,l\geqslant 0}} a_{kl}(1-x)^k(1+x)^l, \qquad \text{with all } a_{kl}\geqslant 0 \text{ or all } a_{kl}\leqslant 0, \tag{1}$$

have been introduced and studied by G. G. Lorentz [1]; we shall call them polynomials with positive or negative (more exactly non-negative or non-positive) coefficients, respectively, or simply Lorentz polynomials. Their set will be denoted by L^+ and L^-, respectively; also let $L=L^+\cup L^-$. The representation (1) is not unique, since multiplying by

$$1=\left(\frac{1-x}{2}+\frac{1+x}{2}\right)^s=2^{-s}\sum_{j=0}^{s}\binom{s}{j}(1-x)^j(1+x)^{s-j} \qquad (s=1,2,...), \tag{2}$$

we obtain other representations. Among all of the representations (1) of a fixed polynomial $p(x)\in L$, consider those for which m is the least possible value. This will be called the Lorentz degree of $p(x)$, and it will be denoted by $d(p)$. If Π_n denotes the set of polynomials of degree at most n, then obviously, $p(x)\in \Pi_n\setminus\Pi_{n-1}$ implies

$$d(p)\geqslant n. \tag{3}$$

The representation (1) of a $p(x)\in L$ with $m=d(p)$ is still not unique, since terms in (1) with $k+l<m$ can be multiplied by (2) with $s=m-k-l\geqslant 1$, resulting in a new representation. However, by this method each representation can be transformed into

$$p(x)=\sum_{k=0}^{d(p)} a_k(1-x)^k(1+x)^{d(p)-k}, \qquad \text{all } a_k\geqslant 0 \text{ or all } a_k\leqslant 0, \tag{4}$$

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and this is already unique, as is easily seen. We shall call (4) the Lorentz representation of $p(x)$.

Our primary concern here is the structural characterization of the set L. This is expressed in the following

THEOREM 1. *A polynomial not identically zero belongs to L if and only if it has no roots in the interval* $(-1, 1)$.

The "only if" part of the statement is trivial: a Lorentz polynomial not identically zero cannot have zeros in $(-1, 1)$. The "if" part is more difficult and it will follow from Theorem 3 below, which can be considered as a quantitative version of Theorem 1.

Remark. Of course, the notion of Lorentz polynomials can be introduced on any finite interval (a, b): this is the set of polynomials $p(x)$ representable in the form

$$p(x) = \sum_{k=0}^{d(p)} a_k(b-x)^k(x-a)^{d(p)-k}, \qquad \text{all} \quad a_k \geqslant 0 \quad \text{or} \quad \leqslant 0.$$

This will be a Lorentz polynomial in any $(c, d) \subset (a, b)$, since substituting

$$b-x = \frac{b-c}{d-c}(d-x) + \frac{b-d}{d-c}(x-c), \qquad x-a = \frac{c-a}{d-c}(d-x) + \frac{d-a}{d-c}(x-c)$$

$$\left(\text{where } \frac{b-c}{d-c}, \frac{b-d}{d-c}, \frac{c-a}{d-c}, \frac{d-a}{d-c} \text{ are non-negative}\right),$$

we get a representation of $p(x)$ on (c, d) with non-negative or non-positive coefficients. This also shows that the Lorentz degree of a polynomial on a subinterval cannot be higher than that on the original interval.

At first we settle the problem of Lorentz representation of quadratic polynomials. In this case we get sharp estimates for the Lorentz degree.

THEOREM 2. (i) *If a quadratic polynomial $p(x)$ has no roots in the open complex unit circle then $d(p) = 2$.*

(ii) *If the roots of a quadratic polynomial $p(x)$ are on the ellipse $y^2 = \varepsilon^2(1-x^2)$ $(0 < \varepsilon < 1,\ |x| < 1)$ then*

$$\frac{1}{\varepsilon^2} \leqslant d(p) < \frac{2}{\varepsilon^2} + 1. \tag{5}$$

Proof. (i) This statement is equivalent to an observation of G. G. Lorentz (see T. Scheick [3]).

(ii) Evidently, we may assume that the leading coefficient of $p(x)$ is 1. Then

$$p(x)=x^2+2\alpha x+\alpha^2+\varepsilon^2(1-\alpha^2) \qquad (|\alpha|<1). \tag{6}$$

We look for a representation

$$p(x)=\sum_{l=0}^{m} a_{lm}(1-x)^l(1+x)^{m-l}, \qquad \text{all } a_{lm}\geqslant 0.$$

Let $u=(1-x)/(1+x)$, then

$$\begin{aligned}2^m \sum_{l=0}^{m} a_{lm}u^l &= (1+u)^{m-2}\{(1-u)^2+2\alpha(1-u^2)\\ &\quad + [\alpha^2+\varepsilon^2(1-\alpha^2)](1+u)^2\}\\ &= (1+u)^{m-2}\{[(1-\alpha)^2+\varepsilon^2(1-\alpha^2)]\,u^2\\ &\quad - 2(1-\varepsilon^2)(1-\alpha^2)\,u+[(1+\alpha)^2+\varepsilon^2(1-\alpha^2)]\}.\end{aligned}$$

Thus

$$\begin{aligned}2^m l!\, a_{lm} &= [(1+\alpha)^2+\varepsilon^2(1-\alpha^2)]\binom{m-2}{l}\, l!\\ &\quad - 2(1-\varepsilon^2)(1-\alpha^2)\binom{l}{1}\binom{m-2}{l-1}(l-1)!\\ &\quad + [(1-\alpha)^2+\varepsilon^2(1-\alpha^2)]\binom{l}{2}\,2\binom{m-2}{l-2}(l-2)!,\end{aligned}$$

i.e.,

$$\begin{aligned}b_{lm} := \frac{2^m l!(m-l)!}{(m-2)!}\, a_{lm} &= (m-l)(m-l-1)[(1+\alpha)^2+\varepsilon^2(1-\alpha^2)]\\ &\quad - 2(m-l)\, l(1-\varepsilon^2)(1-\alpha^2)+l(l-1)[(1-\alpha)^2+\varepsilon^2(1-\alpha^2)]\\ &= (m-2l)^2+2(m-1)(m-2l)\,\alpha+m(m-1)[(1-\alpha^2)\,\varepsilon^2+\alpha^2]-m\\ &\qquad (l=0, ..., m).\end{aligned} \tag{7}$$

Hence the coefficients a_{0m} and a_{mm} are always positive (independently of the choice of $m\geqslant 2$), namely,

$$\frac{2^m}{m(m-1)}\, a_{0m}=(1+\alpha)^2+\varepsilon^2(1-\alpha^2),$$

$$\frac{2^m}{m(m-1)}\, a_{mm}=(1-\alpha)^2+\varepsilon^2(1-\alpha^2).$$

Thus in what follows we will assume that $1 \leqslant l \leqslant m-1$. Let $m \geqslant 2/\varepsilon^2$, then we get from (7)

$$\begin{aligned} b_{lm} &\geqslant (m-2l)^2 + 2(m-1)(m-2l)\,\alpha + (m-1)(m-2)\,\alpha^2 + m - 2 \\ &\geqslant (m-2l)^2 - \frac{m-1}{m-2}(m-2l)^2 + m - 2 \\ &= \frac{4(m-l-1)(l-1)}{m-2} \geqslant 0 \qquad (l = 1, \ldots, m-1). \end{aligned}$$

This proves the upper estimate in (5). To see the lower estimate, let $m < 1/\varepsilon^2$. Then (7) yields

$$\begin{aligned} b_{lm} &< (m-2l)^2 + 2(m-1)(m-2l)\,\alpha + (m-1)^2\,\alpha^2 - 1 \\ &= [m-2l+(m-1)\,\alpha]^2 - 1 \leqslant 0 \end{aligned}$$

provided l is chosen such that $|m-2l+(m-1)\,\alpha| \leqslant 1$. This proves the lower estimate in (5).

The following two examples show that both estimates in (5) are sharp.

EXAMPLE 1. Let $0 < \varepsilon < 1$ be such that $1/\varepsilon^2$ is an integer. Then for the quadratic polynomial

$$p_1(x) = x^2 + 2\,\frac{3\varepsilon^2 - 1}{1-\varepsilon^2}\,x + \frac{8\varepsilon^4 - 5\varepsilon^2 + 1}{1-\varepsilon^2}$$

with roots on the ellipse $y^2 = \varepsilon^2(1-x^2)$, we have $d(p_1) = 1/\varepsilon^2$. Indeed, we get from (7) with $m = 1/\varepsilon^2$

$$\begin{aligned} b_{lm} &= \left[m - 2l + (m-1)\,\frac{3\varepsilon^2-1}{1-\varepsilon^2}\right]^2 \\ &\quad + m(m-1)\,\frac{8\varepsilon^4 - 5\varepsilon^2 + 1}{1-\varepsilon^2} - m - (m-1)^2\,\frac{(3\varepsilon^2-1)^2}{(1-\varepsilon^2)^2} \\ &= (2l-3)^2 + 8 - \frac{6}{\varepsilon^2} + \frac{1}{\varepsilon^4} - 9 + \frac{6}{\varepsilon^2} - \frac{1}{\varepsilon^4} \\ &= (2l-3)^2 - 1 \geqslant 0 \qquad (l = 0, \ldots, m), \end{aligned}$$

i.e., $d(p_1) \leqslant 1/\varepsilon^2$, and by (5), $d(p_1) = 1/\varepsilon^2$.

EXAMPLE 2. Let $0<\varepsilon<1$ be such that $2/\varepsilon^2$ is an integer. Then for the quadratic polynomial

$$p_2(x)=x^2-2\frac{2-3\varepsilon^2}{(1-\varepsilon^2)(2-\varepsilon^2)}x+\frac{-\varepsilon^8+5\varepsilon^6+\varepsilon^4-8\varepsilon^2+4}{(1-\varepsilon^2)(2-\varepsilon^2)^2}$$

with roots on the ellipse $y^2=\varepsilon^2(1-x^2)$, we have $d(p_2)=2/\varepsilon^2$. Indeed, we get from (7) with $m=2/\varepsilon^2-1$

$$b_{1m}=(m-1)\left[\frac{2-5\varepsilon^2}{\varepsilon^2}-2\frac{(2-3\varepsilon^2)^2}{\varepsilon^2(1-\varepsilon^2)(2-\varepsilon^2)}+\frac{-\varepsilon^8+5\varepsilon^6+\varepsilon^4-8\varepsilon^2+4}{\varepsilon^2(1-\varepsilon^2)(2-\varepsilon^2)}\right.$$
$$=-\frac{2\varepsilon^4}{2-\varepsilon^2}<0,$$

i.e., $d(p_2)\geqslant 2/\varepsilon^2$, and by (5), $d(p_2)=2/\varepsilon^2$.

We now turn to the estimate of Lorentz degree of polynomials of higher degree. As we shall see, the results are less complete than for quadratic polynomials. Let $\varphi(x)$ be a positive continuous function in $(-1,1)$ and denote

$$D(\varphi)=\{z=x+iy \mid |y|<\varphi(x),\ |x|<1\}$$

the domain of the complex plane determined by it. Also let

$$L_n(\varphi)=\{p(x)\mid p\in \Pi_n,\ p(z)\neq 0 \text{ in } D(\varphi)\}$$

and

$$d_n(\varphi)=\sup_{p\in L_n(\varphi)} d(p).$$

THEOREM 3. *If*

$$1\geqslant\varepsilon:=\inf_{|x|<1}\frac{\varphi(x)}{\sqrt{1-x^2}}>0 \tag{8}$$

then

$$\frac{c_1 n}{2\varepsilon^{4/3}}\leqslant c_1 n\cdot\max\left\{\frac{1-a^2}{\varepsilon^2},\frac{1}{\varepsilon(\varepsilon+\sqrt{1-a^2})}\right\}\leqslant d_n(\varphi)\leqslant\frac{c_2 n}{\varepsilon^2}, \tag{9}$$

where $|a|\leqslant 1$ is a point where the infimum in (8) is attained.[1]

[1] In what follows $c_1, c_2, \ldots$ will denote absolute positive constants.

Remarks. 1. Theorem 3 implies the missing part of Theorem 1. Indeed, if $p \in \Pi_n$ has no roots in $(-1, 1)$ then for sufficiently small $\varphi_0(x) \equiv \varepsilon > 0$ it has no roots in $D(\varphi_0)$, and thus $p \in L_n(\varphi_0)$, i.e., $p \in L$.

2. In some interesting cases the lower estimate in (9) is of the same order of magnitude as the upper estimate; e.g., if

$$\varphi_1(x) = \varepsilon(1-x^2)^\alpha \qquad \text{with} \quad -\infty < \alpha \leqslant \tfrac{1}{2} \qquad (\text{when } a = 0), \tag{10}$$

or

$$\varphi_2(x) = \varepsilon\sqrt{1-x^2} + 1 - x^2 \qquad (\text{when } a = \pm 1).$$

However, if

$$\varphi_3(x) = \varepsilon^{2/3}(1 - \sqrt{1-\varepsilon^{2/3}} \cdot x)$$

then $a = \sqrt{1-\varepsilon^{2/3}}$ and thus $d_n(\varphi_3) \geqslant c_1 n/\varepsilon^{4/3}$, which is much smaller than the upper estimate. (It is easily seen that the latter estimate is valid for any $\varphi(x)$ satisfying (8), as stated in (9).)

Conjecture. In general, the lower estimate in (9) can be improved to $d_n(\varphi) \geqslant c_1 n/\varepsilon^2$.

Proof of Theorem 3. First we prove the upper estimate in (9). Let $p(x) \in L_n(\varphi)$, then it can be written in the form

$$p(x) = c \prod_{k=1}^{K} (x - a_k) \prod_{k=1}^{N} (x^2 + 2\alpha_k x + \beta_k),$$

where $K + 2N \leqslant n$, $a_k \in \mathbb{R} \setminus (-1, 1)$ $(k = 1, ..., K)$ and

$$\sqrt{\beta_k - \alpha_k^2} \geqslant \varphi(-\alpha_k) \geqslant \varepsilon\sqrt{1-\alpha_k^2} \qquad (k = 1, ..., N). \tag{11}$$

Here the left side inequality follows from the fact that $p(x)$ has no roots in $D(\varphi)$, and the right side inequality is a consequence of (8).

Our method of proof is to represent each factor of $p(x)$ as a Lorentz polynomial, and then multiplying these we get a representation of $p(x)$. The linear factors of $p(x)$ can be written as

$$x - a_k = -\frac{a_k + 1}{2}(1 - x) - \frac{a_k - 1}{2}(1 + x),$$

which is a Lorentz representation since $|a_k| \geqslant 1$. As for the Lorentz representation of the quadratic factors of $p(x)$, it is clear from (6) and (7) that the greater the constant term in a quadratic polynomial is, the greater are its Lorentz coefficients. Since by (11) $\beta_k \geqslant \varepsilon^2(1 - \alpha_k^2) + \alpha_k^2$, the Lorentz degree of $x^2+2\alpha_k x+\beta_k$ is between $1/\varepsilon^2$ and $2/\varepsilon^2+1$, by Theorem 2. Multiplying the Lorentz representations of the linear and quadratic factors we get the upper estimate in (9).

To prove the first lower estimate in (9), let

$$p_3(x)=[(x-a)^2+\varepsilon^2(1-a^2)]^n, \tag{12}$$

where, without loss of generality, we may assume that

$$0\leqslant a<1 \qquad \text{and} \qquad 0<\varepsilon<\frac{1}{4}\sqrt{\frac{1-a}{1+a}}. \tag{13}$$

The roots of this polynomial are $a\pm i\varepsilon\sqrt{1-a^2}=a\pm i\varphi(a)$, by (8) and the definition of a. Hence $p_3(x)\in L_{2n}(\varphi)$. (We shall prove the lower estimates with $2n$ instead of n, which, again, does not restrict generality.) Let

$$p_3(x)=\sum_{k=0}^{d} b_k(1-x)^k(1+x)^{d-k} \qquad (b_k\geqslant 0,\ k=0,\ldots,d)$$

be the Lorentz representation of $p_3(x)$, where $d=d(p_3)$. Using the Cauchy–Schwarz inequality and (13) we get

$$\begin{aligned}
&p_3(a+\varepsilon\sqrt{1-a^2})\,p_3(a+3\varepsilon\sqrt{1-a^2})\\
&\quad=\sum_{k=0}^{d} b_k(1-a-\varepsilon\sqrt{1-a^2})^k(1+a+\varepsilon\sqrt{1-a^2})^{d-k}\\
&\qquad\times\sum_{k=0}^{d} b_k(1-a-3\varepsilon\sqrt{1-a^2})^k(1+a+3\varepsilon\sqrt{1-a^2})^{d-k}\\
&\quad\geqslant\Big\{\sum_{k=0}^{d} b_k[(1-a-\varepsilon\sqrt{1-a^2})(1-a-3\varepsilon\sqrt{1-a^2})]^{k/2}\\
&\qquad\times[(1+a+\varepsilon\sqrt{1-a^2})(1+a+3\varepsilon\sqrt{1-\alpha^2})]^{(d-k)/2}\Big\}^2\\
&\quad=\Big\{\sum_{k=0}^{d} b_k(1-a-2\varepsilon\sqrt{1-a^2})^k(1+a+2\varepsilon\sqrt{1-a^2})^{d-k}\\
&\qquad\times\left[1-\frac{\varepsilon^2(1-a^2)}{(1-a-2\varepsilon\sqrt{1-a^2})^2}\right]^{k/2}\left[1-\frac{\varepsilon^2(1-a^2)}{(1+a+2\varepsilon\sqrt{1-a^2})^2}\right]^{(d-k)/2}\Big\}^2\\
&\quad\geqslant\left(1-\frac{8\varepsilon^2}{1-a}\right)^d\Big\{\sum_{k=0}^{d} b_k(1-a-2\varepsilon\sqrt{1-a^2})^k(1+a+2\varepsilon\sqrt{1-a^2})^{d-k}\Big\}^2\\
&\quad\geqslant\exp\left(-\frac{16\varepsilon^2 d}{1-a}\right)p_3(a+2\varepsilon\sqrt{1-a^2})^2.
\end{aligned}$$

Hence and from (12) we get

$$\exp\frac{16\varepsilon^2 d}{1-a} \geqslant \frac{p_3(a+2\varepsilon\sqrt{1-a^2})^2}{p_3(a+\varepsilon\sqrt{1-a^2})\,p_3(a+3\varepsilon\sqrt{1-a^2})}$$
$$= \frac{[5\varepsilon^2(1-a^2)]^{2n}}{[2\varepsilon^2(1-a^2)]^n[10\varepsilon^2(1-a^2)]^n}$$
$$= (\tfrac{5}{4})^n \geqslant e^{n/5},$$

i.e.,

$$d \geqslant \frac{(1-a)\,n}{80\varepsilon^2} \geqslant \frac{(1-a^2)\,n}{160\varepsilon^2}.$$

To prove the second lower estimate in (9) we need a lemma.

LEMMA 1. *We have* (*for any* φ)

$$d_{2n}(\varphi) \geqslant n \sup_{|a|<1} \frac{1-a^2}{\varphi(a)[\varphi(a)+1-a^2]} \qquad (n=1, 2, ...).$$

Proof. Let $0 \leqslant a < 1$ and consider

$$p_4(x) = [(x-a)^2 + \varphi(a)^2]^n \in L_{2n}(\varphi). \tag{14}$$

Let $a < b \leqslant 1$, then $p_4(x)$ has no root on $(-1, b]$, and Theorem 1 used on this interval yields that $p_4(x)$ is a Lorentz polynomial on $(-1, b)$, with Lorentz degree $\bar{d}(p_4) \leqslant d(p_4)$ (see the remark following Theorem 1). Let the Lorentz expansion of $p_4(x)$ on $(-1, b)$ be

$$p_4(x) = \sum_{k=0}^{\bar{d}(p_4)} a_k(b-x)^k(1+x)^{\bar{d}(p_4)-k} \qquad (a_k \geqslant 0, k=0, ..., \bar{d}(p_4)).$$

Since

$$p_4(b) = a_0(1+b)^{\bar{d}(p_4)} \qquad \text{and} \qquad p_4'(b) = (a_0\bar{d}(p_4) - a_1)(1+b)^{\bar{d}(p_4)-1},$$

we obtain from (14)

$$d(p_4) \geqslant \bar{d}(p_4) \geqslant (1+b)\frac{p_4'(b)}{p_4(b)} = \frac{2(1+b)\,n(b-a)}{(b-a)^2+\varphi(a)^2}.$$

Now if $a + \varphi(a) \leqslant 1$ then let $b = a + \varphi(a)$; hence

$$d(p_4) \geqslant \frac{(1+b)\,n}{\varphi(a)} \geqslant \frac{n(1-a^2)}{\varphi(a)[\varphi(a)+1-a^2]}.$$

If $a+\varphi(a)>1$ then let $b=1$; hence

$$d(p_4)\geqslant\frac{4n(1-a)}{(1-a)^2+\varphi(a)^2}\geqslant\frac{2n(1-a^2)}{\varphi(a)[\varphi(a)+1-a^2]}.$$

The proof in case $-1<a<0$ is similar. ∎

Now in order to prove the second lower estimate in (9), let a in Lemma 1 be such that $\varphi(a)=\varepsilon\sqrt{1-a^2}$. This substitution gives the desired result. The proof of Theorem 3 is complete. ∎

We could not give as exact estimates for polynomials of degree n as for quadratic polynomials. The reason is that our method of multiplying Lorentz representations to get that of the product polynomial generates a certain loss in the degree. Namely, there exist polynomials such that in the obvious inequality

$$d(pq)\leqslant d(p)+d(q)\qquad (p, q\in L)$$

the strict inequality holds. This can be seen from the following

EXAMPLE 3. Let

$$p(x)=1-x+2(1+x)\qquad\text{and}\qquad q(x)=2(1-x)^2-(1-x^2)+2(1+x)^2.$$

Then $d(p)=1$, $d(q)\geqslant 3$, but since

$$p(x)\,q(x)=2(1-x)^3+3(1-x)^2(1+x)+4(1+x)^3,$$

we have $d(pq)=3<d(p)+d(q)$.

Condition (8) in Theorem 3 permits the function $\varphi(x)=\varepsilon\sqrt{1-x^2}$ as a borderline case. The following theorem shows that when (8) is violated (e.g., the domains (10) with $\alpha>\frac{1}{2}$), the situation is completely different.

THEOREM 4. *If at least one of the conditions*

$$\lim_{x\to-1+0}\frac{\varphi(x)}{\sqrt{1+x}}=0\qquad\text{and}\qquad\lim_{x\to1-0}\frac{\varphi(x)}{\sqrt{1-x}}=0\tag{15}$$

holds then $d_n(\varphi)=\infty$ $(n=2, 3, ...)$.

Proof. Assume, e.g., that the second condition in (15) holds. By Lemma 1 we get

$$d_{2n}(\varphi)\geqslant n\bigg/\frac{\varphi(a)}{\sqrt{1-a^2}}\left\{\frac{\varphi(a)}{\sqrt{1-a^2}}+\sqrt{1-a^2}\right\}\qquad (|a|<1).$$

Letting $a \to 1-$ and using (15), we get $d_{2n}(\varphi) = \infty$. This obviously yields $d_n(\varphi) = \infty$ for all $n \geqslant 2$. ∎

We now want to characterize those $\varphi(x)$ for which we have equalities in (3) whenever $p(x) \in L_n(\varphi)$. Let C be the open unit circle in the complex plane.

THEOREM 5. (i) *If* $D(\varphi) \supseteqq C$ *then*

$$d_n(\varphi) = n \tag{16}$$

for all $n = 1, 2, \ldots$.

(ii) *If* (16) *holds for some* $n \geqslant 2$ *then* $D(\varphi) \supseteqq C$.

Proof. (i) By assumption, all the roots of any $p(x) \in L_n(\varphi)$ are on, or outside of, the complex unit circle. Thus (16) follows from Theorem 2(i).

To prove (ii) we need the following

LEMMA 2. *If* $p(x) \in L$ *then*

$$|p(1)| \leqslant e^{(1-x)d(p)} |p(x)| \qquad (0 \leqslant x \leqslant 1).$$

Remark. Of course, a similar statement is true for $p(-1)$ (then $1-x$ is replaced by $1+x$, and $-1 \leqslant x \leqslant 0$ in Lemma 2).

Proof of Lemma 2. Starting from the representation (4) we get

$$p(1) = a_0 2^{d(p)} \leqslant \left(\frac{2}{1+x}\right)^{d(p)} \sum_{k=0}^{d(p)} a_k (1-x)^k (1+x)^{d(p)-k}$$
$$\leqslant (2-x)^{d(p)} p(x) \leqslant e^{(1-x)d(p)} p(x) \qquad (0 \leqslant x \leqslant 1)$$

(assuming that $p(x) > 0$ in $(-1, 1)$). ∎

Now it suffices to prove Theorem 5(ii) when n is even. Consider the polynomial (14). Let $0 < h < 1$; then by Lemma 2 and (16)

$$\frac{p_4(1)}{p_4(h)} = \left[\frac{(1-a)^2 + \varphi(a)^2}{(h-a)^2 + \varphi(a)^2}\right]^n = \left[1 + \frac{(1-h)(1+h-2a)}{(h-a)^2 + \varphi(a)^2}\right]^n$$
$$\leqslant e^{(1-h)d(p_4)} = e^{(1-h)n},$$

i.e.,

$$\frac{1+h-2a}{(h-a)^2 + \varphi(a)^2} \leqslant \frac{e^{1-h} - 1}{1-h}.$$

Now let $h \to 1-$, then

$$\frac{2(1-a)}{(1-a)^2+\varphi(a)^2} \leqslant 1,$$

or $\varphi(a)^2 \geqslant 1-a^2$ $(0 \leqslant a < 1)$. By reason of symmetry, the same inequality holds for $-1 < a < 0$. This proves that $C \subseteqq D(\varphi)$. ▮

To characterize those *individual* polynomials $p(x) \in L$ for which

$$d(p) = \deg p, \tag{17}$$

is a more difficult problem. Let $z_1, z_2, \ldots, z_n$ be the roots of $p(x)$.

PROPOSITION. *A necessary condition for* (17) *to hold is that*

$$|z_1 z_2 \cdots z_n| \geqslant 1. \tag{18}$$

Namely, if, e.g.,

$$p(x) = x^n + \cdots + z_1 \cdots z_n = \sum_{k=0}^{n} a_k (1-x)^k (1+x)^{n-k}, \qquad \text{all } a_k \geqslant 0,$$

then comparing the coefficients of x^n we get

$$z_1 \cdots z_n = p(0) = \sum_{k=0}^{n} a_k \geqslant \sum_{k=0}^{n} (-1)^k a_k = 1.$$

However, (18) is only necessary for (17). Even $z_1 \cdots z_n$ can be arbitrarily large while $d(p) > \deg p$. This will be seen from the following

EXAMPLE 4. For

$$p_5(x) = (x+a)(x^2+\tfrac{1}{4}) \qquad (a \text{ arbitrary})$$

we have $d(p_5) \geqslant 4$. Namely,

$$p_5(x) = \frac{5(a-1)}{32}(1-x)^3 + \frac{11-a}{32}(1-x)^2(1+x)$$
$$- \frac{11+a}{32}(1-x)(1+x)^2 + \frac{5(a+1)}{32}(1+x)^3$$

and here the signs of the coefficients are never the same, whatever the value of a is.

On the other hand, $p(x)$ may have an arbitrarily small (complex) root such that (17) still holds:

EXAMPLE 5. Given $n \geqslant 1$, for the polynomial

$$p_6(x) = (x+a)^n \left(x^2 + \frac{1}{n}\right),$$

we will have $d(p_6) = n+2$, provided a is large enough.

To verify this, let

$$\sum_{k=0}^{n+2} a_k (1-x)^k (1+x)^{n+2-k} = (x+a)^n \left(x^2+\frac{1}{n}\right) = \frac{a^n}{n} + a^{n-1}x + \sum_{j=2}^{n+2} \left[\binom{n}{j-2} a^2 + \binom{n}{j}\frac{1}{n}\right] x^j a^{n-j}.$$

Applying the substitution $u = (1-x)/(1+x)$ and multiplying by $(1+u)^{n+2}$ we get

$$2^{n+2} \sum_{k=0}^{n+2} a_k u^k = \frac{a^n}{n}(1+u)^{n+2} + a^{n-1}(1+u)^{n+1}(1-u) + \sum_{j=2}^{n+2} \left[\binom{n}{j-2} a^2 + \binom{n}{j}\frac{1}{n}\right] [1+u)^{n+2-j}(1-u)^j a^{n-j}.$$

Hence

$$2^{n+2} a_k = \frac{a^n}{n}\binom{n+2}{k} + a^{n-1}\left[\binom{n+1}{k} - \binom{n+1}{k-1}\right] + \sum_{j=2}^{n+2} \left[\binom{n}{j-2} a^2 + \binom{n}{j}\frac{1}{n}\right] a^{n-j} \sum_{l=0}^{k} (-1)^l \binom{j}{l}\binom{n+2-j}{k-l}.$$

Thus a_k is a polynomial of degree n of the variable a. The leading coefficient of this polynomial is (for $2 \leqslant k \leqslant n+2$)

$$\begin{aligned}
&\frac{1}{n}\binom{n+2}{k} + \binom{n}{k} - 2\binom{n}{k-1} + \binom{n}{k-2} \\
&= \frac{k!(n+2-k)!}{n!}\left[\frac{(n+2)(n+1)}{n}\right. \\
&\quad \left. + (n-k+2)(n-k+1) - 2k(n-k+2) + k(k-1)\right] \\
&= \frac{k!(n-k+2)!}{n!}\left[\frac{(n+2)(n+1)}{n} + (n-2k)^2 + 4(n-2k) - n + 2\right] \\
&\geqslant \frac{n+2}{n}\frac{k!(n-k+2)!}{n!} > 0.
\end{aligned}$$

The same coefficients for a_0 and a_1 are $1+1/n$ and $n+2/n$, respectively, thus positive. Hence if a is large enough then all a_k will be positive.

Some properties of polynomials carried over to Lorentz polynomials show an improvement in the order of estimations. A remarkable property of polynomials is the following theorem of Schur (cf. Lorentz [2, p. 41]): if $p(x) \in \Pi_n$ then

$$\|p(x)\| \leqslant (n+1)\|p(x)\sqrt{1-x^2}\|, \tag{19}$$

where $\|\cdot\|$ denotes the supremum norm in $[-1, 1]$. For Lorentz polynomials we have the following sharper result.

THEOREM 6. *Let $\alpha > 0$ be a real number and $n \geqslant 1$ an integer. Then*

$$\sup_{0 \not\equiv p \in L, d(p) \leqslant n} \frac{\|p(x)\|}{\|p(x)(1-x^2)^\alpha\|} = \frac{(n+2\alpha)^{n+2\alpha}}{(4\alpha)^\alpha (n+\alpha)^{n+\alpha}} < \left[\frac{e}{4\alpha}(n+2\alpha)\right]^\alpha$$

and the supremum is attained if and only if $p(x) = c(1 \pm x)^n$ $(c \neq 0)$.

Proof. If $|y| \leqslant x_1 := n/(n+2\alpha)$ then

$$\frac{\|p(y)\|}{\|p(x)(1-x^2)^\alpha\|} \leqslant \frac{1}{(1-y^2)^\alpha} \leqslant \frac{1}{(1-x_1^2)^\alpha}$$
$$= \frac{(n+2\alpha)^{2\alpha}}{(4\alpha)^\alpha (n+\alpha)^\alpha} < \frac{(n+2\alpha)^{n+2\alpha}}{(4\alpha)^\alpha (n+\alpha)^{n+\alpha}}. \tag{20}$$

If $x_1 < |y| \leqslant 1$, say $x_1 < y \leqslant 1$, then we get (taking $0^0 = 1$)

$$(1-y)^k (1+y)^{n-k} \leqslant \|(1-x)^k (1+x)^{n-k}\| = \frac{2^n k^k (n-k)^{n-k}}{n^n}$$
$$= \frac{k^k (n-k)^{n-k} (n+2\alpha)^n}{\alpha^k n^n (n+\alpha)^{n-k}} (1-x_1)^k (1+x_1)^{n-k}$$
$$\leqslant \left[\frac{k(n+\alpha)}{\alpha(n-k)}\right]^k \left(\frac{n+2\alpha}{n+\alpha}\right)^n \times (1-x_1)^k (1+x_1)^{n-k}$$
$$\leqslant \left(\frac{n+2\alpha}{n+\alpha}\right)^n (1-x_1)^k (1+x_1)^{n-k} \left(0 \leqslant k \leqslant \frac{\alpha n}{n+2\alpha}\right). \tag{21}$$

On the other hand, since $(1-x)^k (1+x)^{n-k}$ is monotone decreasing in $[1-2k/n, 1]$, we get

$$(1-y)^k (1+y)^{n-k} < (1-x_1)^k (1+x_1)^{n-k} \qquad \left(\frac{\alpha n}{n+2\alpha} < k \leqslant n\right). \tag{22}$$

Now if $p(x)=\sum_{k=0}^{n} a_k(1-x)^k(1+x)^{n-k}$ $(a_k \geqslant 0, k=0, ..., n)$, then by (21) and (22) we obtain

$$p(y)=\left(\frac{n+2\alpha}{n+\alpha}\right)^n p(x_1)=\frac{(n+2\alpha)^{n+2\alpha}}{(4\alpha)^\alpha(n+\alpha)^{n+\alpha}} p(x_1)(1-x_1^2)^\alpha$$
$$\leqslant \frac{(n+2\alpha)^{n+2\alpha}}{(4\alpha)^\alpha(n+\alpha)^{n+\alpha}} \|p(x)(1-x^2)^\alpha\|.$$

This together with (20) shows that

$$\frac{\|p(x)\|}{\|p(x)(1-x^2)^\alpha\|} \leqslant \frac{(n+2\alpha)^{n+2\alpha}}{(4\alpha)^\alpha(n+\alpha)^{n+\alpha}}. \tag{23}$$

Here for $p(x)=c(1 \pm x)^n$ $(c \neq 0)$, the equality holds. Conversely, if we have equality in (23), then (since in (20) the strict inequality holds) we must have equalities in (21) for a suitable $y \in [x_1, 1]$, which is possible only if $k=0$, i.e., $p(x)=a_0(1+x)^n$ $(a_0 \neq 0)$. The other case (i.e., $y \in [-1, -x_1]$) yields $p(x)=a_n(1-x)^n$ $(a_n \neq 0)$.

Remarks. Applying Theorem 6 with $\alpha=\frac{1}{2}$ we get

$$\|p(x)\| \leqslant \sqrt{\frac{e}{2}(n+1)}\, \|p(x)\sqrt{1-x^2}\| \qquad (p \in L, n=d(p)),$$

which is better than (19).

Comparing Theorem 6 with Theorem 2 we can see that if $p(x) \in \Pi_n$ has all its roots outside the open unit circle then (23) holds. A direct proof of this statement would be interesting.

Another application of Lorentz polynomials is the estimation of the derivatives. As we shall see, significant improvement in the order of Bernstein and Markov type inequalities will be achieved. In what follows $\|\cdot\|_{L^q}$ means the L^q norm on $[-1, 1]$.

THEOREM 7. *If $p(x) \in L$ then*

$$\|p^{(r)}(x)\|_{L^q} \leqslant K_r 2^{1/q} d(p)^r \|p(x)\|_{L^q} \qquad (0<q \leqslant \infty, r=1, 2, ...),$$

where $K_r>0$ depends only on r.

(For convenience, we use the norm notation even when $0<q<1$.)

Proof. The proof is based on the following estimate of [4]: if the roots of $p(x) \in L$ are in the set $\mathbb{R} \setminus (-1, 1)$ and $p(x)>0$ in $(-1, 1)$ then

$$|p^{(r)}(x)| \leqslant K_r d(p)^r p\left(x \pm \frac{r}{2d(p)}\right) \qquad (|x|<1, r=1, ..., d(p)), \tag{24}$$

where the $+$ or $-$ sign is to be taken according as $x<0$ or $x>0$ (either can be taken if $x=0$). Using the decomposition (4) and applying (24) for the polynomials $a_k(1-x)^k(1+x)^{d(p)-k}$ $(k=0, ..., d(p))$ and adding the resulting inequalities we obtain that (24) is true for any $p(x) \in L$. Raising to the qth power in (24) and integrating we get the statement by noticing that the expression $x \pm r/2d(p)$ assumes the same value at most twice.

THEOREM 8. *If $p(x) \in L$ then*

$$\|p'(x)\sqrt{1-x^2}\|_{L^q} \leqslant c_4 5^{1/q} \sqrt{d(p)}\, \|p(x)\|_{L^q} \qquad (0<q\leqslant\infty). \tag{25}$$

Proof. The case $q=\infty$ was proved by Lorentz [1, Theorem B], so suppose $0<q<\infty$. We make use of the following inequality of [5] (see Lemma 2): if $0 \leqslant p(x) \in L$ then with a suitable $c_5>0$

$$\Delta(x)\,|p'(x)| \leqslant \max\left\{p(x), p\left(x \pm \frac{1}{2}\Delta(x)\right)\right\} \qquad \left(|x| \leqslant 1-\frac{c_5}{d(p)}\right), \tag{26}$$

where $\Delta(x)=\sqrt{(1-x^2)/d(p)}$. Assume that c_5 is so large that

$$y'_{\pm}(x) \geqslant \frac{1}{2} \qquad \text{and} \qquad |y_{\pm}(x)| \leqslant 1 \qquad \text{if} \quad |x| \leqslant 1-\frac{c_5}{d(p)}, \tag{27}$$

where $y_{\pm}(x) := x \pm \frac{1}{2}\Delta(x)$. Integrating the qth power of (26) over $|x| \leqslant 1-c_5/d(p)$ and making the substitutions $y_{\pm}=y_{\pm}(x)$ in the corresponding integrals we obtain by (27)

$$\int_{|x| \leqslant 1-c_5/d(p)} (|p'(x)|\sqrt{1-x^2})^q\,dx \leqslant 5d(p)^{q/2} \int_{-1}^{1} p(x)^q\,dx. \tag{28}$$

Finally, using Theorem 7 we obtain

$$\int_{1-c_5/d(p) \leqslant |x| \leqslant 1} (|p'(x)|\sqrt{1-x^2})^q\,dx \leqslant \frac{c_6^q}{d(p)^{q/2}} \int_{-1}^{1} |p'(x)|^q\,dx$$
$$\leqslant c_7^q d(p)^{q/2}\, \|p(x)\|_{L^q}^q.$$

This together with (28) yields Theorem 8.

Using the estimate of Theorem 3 we can get Markov and Bernstein type inequalities for Lorentz polynomials in terms of the ordinary degree of the polynomial:

COROLLARY. *Under the conditions of Theorem* 3 *we have*

(i) $\|p^{(r)}(x)\|_{L^q} \leqslant 2^{1/q}\bar{K}_r(n/\varepsilon^2)^r \, \|p(x)\|_{L^q}$,

(ii) $\|p^{(r)}(x)(1-x^2)^{r/2}\|_{L^\infty} \leqslant \bar{\bar{K}}_r(n/\varepsilon^2)^{r/2} \, \|p(x)\|_{L^\infty}$,

(iii) $\|p'(x)\sqrt{1-x^2}\|_{L^q} \leqslant c_8 \, 5^{1/q}(\sqrt{n}/\varepsilon) \, \|p(x)\|_{L^q}$

for all $0 < q \leqslant \infty$ *and* $r = 1, 2, \ldots$.

(i) and (iii) follow from Theorems 7 and 8, respectively, while (ii) is a consequence of the inequality

$$\|p^{(r)}(x)(1-x^2)^{r/2}\|_{L^\infty} \leqslant \bar{\bar{K}}_r' d(p)^{r/2} \, \|p(x)\|_{L^\infty}$$

(cf. Lorentz [1, Theorem B]).

Comparing these inequalities (i)–(iii) with the ordinary Markov–Bernstein type estimates we can see that the exponent of n is half the original.

Note added in proof. After preparing this manuscript the authors discovered that Theorem 1 is not new (see G. Pólya and G. Szegő, Problems and theorems in analysis, Volume II, p. 78, Problem 49). In any case, Theorem 1 is a simple consequence of our main result, Theorem 3.

REFERENCES

1. G. G. LORENTZ, The degree of approximation by polynomials with positive coefficients, *Math. Ann.* **151** (1963), 239–251.
2. G. G. LORENTZ, "Approximation of Functions," Holt, Rinehart & Winston, New York, 1966.
3. J. T. SCHEICK, Inequalities for derivatives of polynomials of special type, *J. Approx. Theory* **6** (1972), 354–358.
4. A. MÁTÉ, Inequalities for derivatives of polynomials with restricted zeros, *Proc. Amer. Math Soc.* **88** (1981), 221–225.
5. T. ERDÉLYI AND J. SZABADOS, Bernstein type inequalities for a class of polynomials, *Acta Math. Hungar.* **51** (1988), in press.

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